# POLYGON CONVEXITY: ANOTHER $O(n)$ TEST

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Abstract. An $n$-gon is defined as a sequence $\mathcal{P} = (V_0, \dots, V_{n-1})$ of $n$ points on the plane. An $n$-gon $\mathcal{P}$ is said to be convex if the boundary of the convex hull of the set $\{V_0, \dots, V_{n-1}\}$ of the vertices of $\mathcal{P}$ coincides with the union of the edges $[V_0, V_1], \dots, [V_{n-1}, V_0]$; if at that no three vertices of $\mathcal{P}$ are collinear then $\mathcal{P}$ is called strictly convex. We prove that an $n$-gon $\mathcal{P}$ with $n \geqslant 3$ is strictly convex if and only if a cyclic shift of the sequence $(\alpha_0, \dots, \alpha_{n-1}) \in [0, 2\pi)^n$ of the angles between the $x$-axis and the vectors $\overrightarrow{V_0V_1}, \dots, \overrightarrow{V_{n-1}V_0}$ is strictly monotone.

A "non-strict" version of this result is also proved.

## 1. Definitions and results

A *polygon* is defined in this paper as any finite sequence of points (or, interchangeably, vectors) on the Euclidean plane $\mathbb{R}^2$; the same definition was used in [5–8]. Let here $\mathcal{P} := (V_0, \dots, V_{n-1})$ be a polygon, which is sequence of $n$ points; such a polygon is also called an $n$-gon. The points $V_0, \dots, V_{n-1}$ are called the *vertices* of $\mathcal{P}$. The smallest value that one may allow for the integer $n$ is 0, corresponding to a polygon with no vertices, that is, to the sequence () of length 0. The segments, or closed intervals,

$$[V_i, V_{i+1}] := \operatorname{conv}\{V_i, V_{i+1}\} \quad \text{for } i \in \{0, \dots, n-1\}$$

are called the *edges* of polygon $\mathcal{P}$, where

$$V_n := V_0.$$

The symbol conv denotes, as usual, the convex hull [9, page 12]. Note that, if $V_i = V_{i+1}$, then the edge $[V_i, V_{i+1}]$ is a singleton set. For any two points $U$ and $V$ in $\mathbb{R}^2$, let $[U, V] := \operatorname{conv}\{U, V\}$, $[U, V) := [U, V] \setminus \{V\}$, and $(U, V) := [U, V] \setminus \{U, V\}$, so that $(U, V) = \operatorname{ri}[U, V]$, the relative interior of $[U, V]$. Let us define the convex hull and dimension of polygon $\mathcal{P}$ as, respectively, the convex hull and dimension of the set of its vertices: $\operatorname{conv}\mathcal{P} := \operatorname{conv}\{V_0, \dots, V_{n-1}\}$ and $\dim \mathcal{P} :=$

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$\dim\{V_0, \dots, V_{n-1}\} = \dim \operatorname{conv} \mathcal{P}$. In general, our terminology corresponds to that in [9].

In the sequel, we also use the notation

$$\overline{k,m} := \{i \in \mathbb{Z}\colon k \leqslant i \leqslant m\},$$

where $\mathbb{Z}$ is the set of all integers; in particular, $\overline{k,m}$ is empty if $m < k$.

Given the above notion of the polygon, a *convex polygon* can be defined as a polygon $\mathcal{P}$ such that the union of the edges of $\mathcal{P}$ coincides with the boundary $\partial \operatorname{conv} \mathcal{P}$ of the convex hull $\operatorname{conv} \mathcal{P}$ of $\mathcal{P}$; cf. e.g. [11, page 5]. Thus, one has

**Definition 1.1.** *A polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is* convex *if*

$$\bigcup_{i \in \overline{0,n-1}} [V_i, V_{i+1}] = \partial \operatorname{conv} \mathcal{P}.$$

Let us emphasize that a polygon in this paper is a sequence and therefore ordered. In particular, even if all the vertices $V_0, \dots, V_{n-1}$ of a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ are the extreme points of the convex hull of $\mathcal{P}$, it does not necessarily follow that $\mathcal{P}$ is convex. For example, consider the points $V_0 = (0,0)$, $V_1 = (1,0)$, $V_2 = (1,1)$, and $V_3 = (0,1)$. Then polygon $(V_0, V_1, V_2, V_3)$ is convex, while polygon $(V_0, V_2, V_1, V_3)$ is not.

**Definition 1.2.** *Let us say that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is*

- locally-ordinary *– if for any $i$ in the set $\overline{0,n-1}$ the vertices $V_i$ and $V_{i+1}$ are distinct;*
- ordinary *– if for any two distinct $i$ and $j$ in $\overline{0,n-1}$ the vertices $V_i$ and $V_j$ are distinct;*
- locally-strict *– if for any $i$ in $\overline{0,n-1}$ the vertices $V_{i-1}$, $V_i$, and $V_{i+1}$ are non-collinear, where $V_{-1} := V_{n-1}$;*
- quasi-strict *– if any two adjacent vertices of $\mathcal{P}$ are not collinear with any other vertex of $\mathcal{P}$ or, more formally, if for any $i \in \overline{0,n-1}$ and any $j \in \overline{0,n-1} \setminus \{i, i \oplus 1\}$ the points $V_i$, $V_{i\oplus 1}$, and $V_j$ are non-collinear, where*

$$i \oplus 1 := \begin{cases} i+1 & \text{if } i \in \overline{0,n-2}, \\ 0 & \text{if } i = n-1; \end{cases}$$

- strict *– if for any three distinct $i$, $j$, and $k$ in $\overline{0,n-1}$ the vertices $V_i$, $V_j$, and $V_k$ are non-collinear;*
- locally-simple *– if for any $i$ in $\overline{0,n-1}$ one has $[V_i, V_{i+1}) \cap [V_{i+1}, V_{i+2}) = \emptyset$, where $V_{n+1} := V_1$;*
- simple *– if for any two distinct $i$ and $j$ in $\overline{0,n-1}$ one has $[V_i, V_{i+1}) \cap [V_j, V_{j+1}) = \emptyset$;*
- locally-ordinarily convex *– if $\mathcal{P}$ is locally-ordinary and convex; similarly can be defined* ordinarily convex, …, simply convex *polygons.*

**Remark 1.3.** *Any (locally-)simple polygon is (locally-)ordinary, since* $\{V_i\}\cap\{V_j\} \subseteq [V_i, V_{i+1}) \cap [V_j, V_{j+1})$.

We shall make use of the following result given in [6]. If $\mathcal{P} = (V_0, \dots, V_{n-1})$ is a polygon, let us refer to any subsequence $(V_{i_0}, \dots, V_{i_{m-1}})$ of $\mathcal{P}$, with $0 \leqslant i_0 < \dots < i_{m-1} \leqslant n-1$, as a *sub-polygon* or, more specifically, as a *sub-$m$-gon* of $\mathcal{P}$.

**Theorem 1.4.** [6, Corollary 1.17] *If* $\mathcal{P} = (V_0, \dots, V_{n-1})$ *is an ordinarily convex polygon, then any sub-polygon of* $\mathcal{P}$ *is so; in particular, then the sub-*$(n-1)$*-gons* $\mathcal{P}^{(i)} := (V_0, \dots, V_{i-1}, V_{i+1}, \dots, V_{n-1})$ *of* $\mathcal{P}$ *are ordinarily convex, for all* $i \in \overline{0, n-1}$.

For a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$, let $x_i$ and $y_i$ denote the coordinates of its vertices $V_i$, so that

$$V_i = (x_i, y_i) \quad \text{for } i \in \overline{0, n-1}. \tag{1}$$

Introduce the determinants

$$\Delta_{\alpha,i,j} := \begin{vmatrix} 1 & x_\alpha & y_\alpha \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{vmatrix} \tag{2}$$

for $\alpha$, $i$, and $j$ in the set $\overline{0, n-1}$. Let then

$$\begin{aligned} a_i &:= \operatorname{sign} \Delta_{i+1,i-1,i} = \operatorname{sign} \Delta_{i-1,i,i+1}; \\ b_i &:= \operatorname{sign} \Delta_{0,i-1,i}; \\ c_i &:= \operatorname{sign} \Delta_{i,0,1} = \operatorname{sign} \Delta_{0,1,i}. \end{aligned}$$

The following theorem is the main result of [7], which provides an $O(n)$ test of the strict convexity of a polygon.

**Theorem 1.5.** [7] *An $n$-gon* $\mathcal{P} = (V_0, \dots, V_{n-1})$ *with* $n \geqslant 4$ *is strictly convex if and only if conditions*

$$\begin{gathered} a_i b_i > 0, \\ a_i b_{i+1} > 0, \\ c_i c_{i+1} > 0 \end{gathered} \tag{3}$$

*hold for all*

$$i \in \overline{2, n-2}.$$

**Proposition 1.6.** [7] *None of the* $3(n-3)$ *conditions in Theorem 1.5 can be omitted without (the "if" part of) Theorem 1.5 ceasing to hold.*

Thus, the test given by Theorem 1.5 is exactly minimal.

**Remark 1.7.** [7] *Adding to the* $3(n-3)$ *conditions* (3) *in Theorem 1.5 the equality* $b_2 = c_2$*, which trivially holds for any polygon (convex or not), one can rewrite* (3) *as the following system of* $3(n-3)+1$ *equations and one inequality:*

$$\begin{aligned}&a_2 = \cdots = a_{n-2}\\ =&b_2 = \cdots = b_{n-2} = b_{n-1}\\ =&c_2 = \cdots = c_{n-2} = c_{n-1} \neq 0.\end{aligned}$$

These results were used in [8].

For any vector $\vec{v} = (x, y) \in \mathbb{R}^2$ with $r := |\vec{v}| := \sqrt{x^2 + y^2} \neq 0$, define the (angle) argument of $\vec{v}$ as usual, by the formula

$$\arg \vec{v} = \psi \iff \big(0 \leqslant \psi < 2\pi \ \&\ x = r\cos\psi \ \&\ y = r\sin\psi\big), \tag{4}$$

so that, for each nonzero vector $\vec{v} \in \mathbb{R}^2$, the "angle" $\arg \vec{v}$ is a uniquely defined number in the interval $[0, 2\pi)$. Moreover,

$$\arg \vec{v} = \begin{cases} \arccos \frac{x}{r} & \text{if } \vec{v} \in H_-,\\ 2\pi - \arccos \frac{x}{r} & \text{if } \vec{v} \in H_+, \end{cases} \tag{5}$$

where arccos is the branch of the inverse function $\cos^{-1}$ with values in the interval $[0, \pi]$ and

$$\begin{aligned} H_- &:= \{(x, y) \in \mathbb{R}^2 \colon y > 0 \text{ or } (y = 0 \ \&\ x > 0)\},\\ H_+ &:= \{(x, y) \in \mathbb{R}^2 \colon y < 0 \text{ or } (y = 0 \ \&\ x < 0)\}; \end{aligned}$$

note that $H_- \cup H_+ = \mathbb{R}^2 \setminus \{\vec{0}\}$ and $H_- \cap H_+ = \emptyset$.

For any locally-ordinary polygon $\mathcal{P} = (V_0, \ldots, V_{n-1})$, introduce also the sequence of the angle arguments the edge-vectors $\overrightarrow{V_0V_1}, \ldots, \overrightarrow{V_{n-1}V_n}$ of $\mathcal{P}$, by the formula

$$\arg \mathcal{P} := (\arg \overrightarrow{V_0V_1}, \ldots, \arg \overrightarrow{V_{n-1}V_n}).$$

For any two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$, let us write

$$\vec{u} < \vec{v} \quad \text{iff} \quad \arg \vec{u} < \arg \vec{v};$$

similarly defined is the relation $>$ on $\mathbb{R}^2 \setminus \{\vec{0}\}$.

**Remark 1.8.** *Let* $\vec{u} = (s, t)$ *and* $\vec{v} = (x, y)$ *be any two vectors in* $\mathbb{R}^2 \setminus \{\vec{0}\}$*. Then, using* (5)*, it is elementary but somewhat tedious to check that*

$$\vec{u} < \vec{v} \iff \begin{cases} \vec{u} \in H_- \ \&\ \vec{v} \in H_+ & \text{or}\\ \vec{u} \in H_- \ \&\ \vec{v} \in H_- \ \&\ \Delta > 0 & \text{or}\\ \vec{u} \in H_+ \ \&\ \vec{v} \in H_+ \ \&\ \Delta > 0, \end{cases} \tag{6}$$

*where*

$$\Delta := \begin{vmatrix} 1 & 0 & 0 \\ 1 & s & t \\ 1 & x & y \end{vmatrix} = \begin{vmatrix} s & t \\ x & y \end{vmatrix} = sy - tx.$$

*Under the additional condition that $\vec{u}$ and $\vec{v}$ are non-collinear, it follows that either $\vec{u} < \vec{v}$ or $\vec{v} < \vec{u}$:*

$$\Delta \neq 0 \implies (\vec{u} < \vec{v} \textit{ or } \vec{v} < \vec{u}).$$

*Note also that*

$$\vec{u} < \vec{v} \implies (y \leqslant 0 \leqslant t \text{ or } \Delta > 0). \tag{7}$$

**Definition 1.9.** *Let us say that a locally-ordinary polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with* $\arg \mathcal{P} =: (\alpha_0, \dots, \alpha_{n-1})$ *is*

- increasing – *if the sequence* $\arg \mathcal{P}$ *is increasing:* $\alpha_0 < \cdots < \alpha_{n-1}$*;*
- decreasing – *if* $\alpha_0 > \cdots > \alpha_{n-1}$*;*
- cyclically increasing *or, briefly,* c-increasing – *if*

  $$\alpha_k < \cdots < \alpha_{n-1} < \alpha_0 < \cdots < \alpha_{k-1},$$

  *for some $k \in \overline{0, n-1}$; (if $k = 0$ then this chain of inequalities is supposed to read simply as $\alpha_0 < \cdots < \alpha_{n-1}$, in which case polygon $\mathcal{P}$ will be increasing);*
- cyclically decreasing *or, briefly,* c-decreasing – *similarly, if*

  $$\alpha_k > \cdots > \alpha_{n-1} > \alpha_0 > \cdots > \alpha_{k-1},$$

  *for some $k \in \overline{0, n-1}$;*
- cyclically strictly monotone *or, briefly,* c-strictly monotone – *if $\mathcal{P}$ is either c-increasing or c-decreasing.*

*The notions of nondecreasing, nonincreasing, c-nondecreasing, c-nonincreasing, and c-monotone polygons are defined similarly, with signs $\leqslant$ and $\geqslant$ replacing $<$ and $>$, respectively.*

For any $k \in \overline{0, n-1}$, define the cyclic shift (or, briefly, c-shift) $\theta^k$ of the sequence $(u_0, \dots, u_{n-1})$ of any objects $u_0, \dots, u_{n-1}$ by the formula

$$(u_0, \dots, u_{n-1})\theta^k := (u_k, \dots, u_{n-1}, u_0, \dots, u_{k-1}).$$

**Remark 1.10.** *For any polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ and any $k \in \overline{0, n-1}$, one has $\arg(\mathcal{P}\theta^k) = (\arg \mathcal{P})\theta^k$. It follows that $\mathcal{P}$ is c-increasing iff $\mathcal{P}$ is a c-shift of an increasing polygon iff a c-shift of $\mathcal{P}$ is increasing; similarly for "decreasing" vs. "c-decreasing" and for other such pairs of terms. Also, all c-shifts preserve the polygon convexity and all properties defined in Definition 1.2 as well as all the "cyclic" properties defined in Definition 1.9: being c-increasing, being c-increasing,....*

For any transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and any polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$, define the corresponding transformation of $\mathcal{P}$ as the polygon $T\mathcal{P} := (TV_0, \dots, TV_{n-1})$. A rotation is any orthogonal (and hence linear) transformation with determinant 1; any rotation can be represented as the linear transformation $R_\alpha$ with matrix $\left[\begin{smallmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{smallmatrix}\right]$ for some real number $\alpha$, so that $R_\alpha\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}\cos\alpha\\ \sin\alpha\end{smallmatrix}\right]$. The reflection is denoted here by $R$ and defined by the formula $\mathbb{R}^2 \ni (x,y) \mapsto R(x,y) := (x,-y)$. Any orthogonal transformation can be represented as $R_\alpha R$ (as well as $RR_\beta$) for appropriate $\alpha$ and $\beta$. A homothetical transformation is understood here as one of the form $\mathbb{R}^2 \ni \vec{v} \to \lambda\vec{v}$ for some $\lambda > 0$.

**Proposition 1.11.** *The properties of being convex; locally-ordinary; ordinary; locally-strict; quasi-strict; strict; locally-simple; and simple are each preserved for any polygon under any nonsingular affine transformation. The properties of being c-increasing, c-decreasing, c-nondecreasing, and c-nonincreasing are each preserved under any rotation or any homothetical transformation or any parallel translation. The properties of being c-increasing and c-decreasing are interchanged under the reflection.*

All the necessary proofs are deferred to Section 2.

The following theorem is one of the main results of this paper.

**Theorem 1.12.** *An $n$-gon with $n \geqslant 3$ is strictly convex iff it is c-strictly monotone.*

**Remark 1.13.** *Any $n$-gon with $n \leqslant 1$ is, trivially, both strictly convex and c-strictly monotone. Any 2-gon is, trivially, strictly convex; however, a 2-gon is c-strictly monotone only if it is locally-ordinary (and hence ordinary). It is easy to see that any strict 3-gon is c-strictly monotone, so that Theorem 1.12 is trivial for $n = 3$. Note also that an $n$-gon is both c-increasing and c-decreasing iff it is locally-ordinary and $n = 2$.*

Theorem 1.12 is complemented by the following proposition, which will also be of use in the proof of Theorem 1.12.

**Proposition 1.14.** *For any $n$-gon $\mathcal{P}$ with $n \geqslant 3$ the following statements are equivalent to one another:*

(I) *$\mathcal{P}$ is ordinary and locally-strictly convex;*
(II) *$\mathcal{P}$ is quasi-strictly convex;*
(III) *$\mathcal{P}$ is strictly convex.*

**Remark.** *The conditions in Proposition 1.14 that $n \geqslant 3$ and $\mathcal{P}$ is ordinary cannot be dropped. Indeed, all 2-gons are strictly convex, but not all of them are ordinary. On the other hand, the polygon $(V_0, V_1, V_2, V_0, V_1, V_2)$ is locally-strictly convex if the points $V_0, V_1, V_2$ are non-collinear, but it is not ordinary and not strictly convex.*

The following theorem is a "non-strict"' counterpart of Theorem 1.12.

**Theorem 1.15.** *For any $n$-gon $\mathcal{P}$ with $n \geqslant 3$ the following statements are equivalent to one another*

(I) *$\mathcal{P}$ is ordinary and locally-simply convex;*
(II) *$\mathcal{P}$ is simply convex;*
(III) *$\mathcal{P}$ is c-monotone and* $\dim \mathcal{P} = 2$.

**Remark.** *None of the following conditions: (i) $n \geqslant 3$, (ii) $\mathcal{P}$ is ordinary, or (iii) $\dim \mathcal{P} = 2$ in Theorem 1.15 can be dropped. Indeed, (i) all 2-gons are simply convex but not all of them are ordinary; (ii) for any three non-collinear points $V_0, V_1, V_2$, the 6-gon $(V_0, V_1, V_2, V_0, V_1, V_2)$ is of dimension 2 and locally-simply convex but not simply convex or c-monotone (or ordinary); (iii) for any three distinct collinear points $V_0, V_1, V_2$, the 3-gon $(V_0, V_1, V_2)$ is c-monotone and ordinary but not simply convex or locally-simply convex (or of dimension 2).*

A suggestion to use c-strict monotonicity to test for polygon convexity was given in [3], without a proof. A result, similar to Theorem 1.15 was presented in [2, Lemma 5 in Section 10.3], with a very brief, heuristic proof.

## 2. Proofs

*Proof of Proposition 1.11.* Suppose that an $n$-gon $\mathcal{P}$ with $(\alpha_0, \dots, \alpha_{n-1}) := \arg(\mathcal{P})$ is c-increasing, that is,

$$(\mathrm{Incr}_k) \qquad \alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1},$$

for some $k \in \overline{0, n-1}$. Let $(\beta_0, \dots, \beta_{n-1}) := \arg(R\mathcal{P})$, the argument sequence of the reflected polygon $R\mathcal{P}$. Then $\beta_i = 2\pi - \alpha_i$ for all $i \neq k$, while $\beta_k = 2\pi - \alpha_k$ if $\alpha_k \neq 0$ and $\beta_k = 0$ if $\alpha_k = 0$. It follows that

$$(\mathrm{Decr}_k) \qquad \beta_k > \dots > \beta_{n-1} > \beta_0 > \dots > \beta_{k-1}$$

if $\alpha_k \neq 0$, and the $\beta_i$'s satisfy $(\mathrm{Decr}_{k\oplus 1})$ if $\alpha_k = 0$, where $k \oplus 1 := k+1$ if $k \in \overline{0, n-2}$ and $k \oplus 1 := 0$ if $k = n-1$.

Similarly, if $\arg \mathcal{P} =: (\beta_0, \dots, \beta_{n-1})$ satisfies condition $(\mathrm{Decr}_k)$ for some $k \in \overline{0, n-1}$, then $(\alpha_0, \dots, \alpha_{n-1}) := \arg(R\mathcal{P})$ satisfies condition $(\mathrm{Incr}_k)$ if $\alpha_{k-1} \neq 0$, and the $\alpha_i$'s satisfy $(\mathrm{Incr}_{k\ominus 1})$ if $\alpha_{k-1} = 0$, where $k \ominus 1 := k-1$ if $k \in \overline{1, n-1}$ and $k \oplus 1 := n-1$ if $k = 0$.

Thus, reflection $R$ interchanges the properties of being c-increasing and c-decreasing.

Let us now verify the preservation of the c-increasing property under any rotation $R_\alpha$. W.l.o.g., $0 \leqslant \alpha < 2\pi$. Suppose again that an $n$-gon $\mathcal{P}$ with $(\alpha_0, \dots, \alpha_{n-1}) := \arg(\mathcal{P})$ satisfies condition $(\mathrm{Incr}_k)$. Then the c-shift $\mathcal{Q} := \mathcal{P}\theta^k := (V_k, \dots, V_{n-1}, V_0, \dots, V_{k-1})$ of polygon $\mathcal{P}$ with $(\beta_0, \dots, \beta_{n-1}) := \arg(\mathcal{Q}) = (\alpha_k, \dots, \alpha_{n-1}, \alpha_0, \dots, \alpha_{k-1})$ is an increasing $n$-gon. Let $(\psi_0, \dots, \psi_{n-1}) := \arg(R_\alpha \mathcal{Q})$. Let $J := \{i \in \overline{0, n-1} \colon \beta_i + \alpha \geqslant 2\pi\}$, and let $j := \min J$ if $J \neq \emptyset$ and $j := n$ if $J = \emptyset$. Then $\psi_i := \beta_i + \alpha$ for $i \in \overline{0, j-1}$ and $\psi_i := \beta_i + \alpha - 2\pi$ for $i \in \overline{j, n-1}$. Hence, the sequence $\arg(R_\alpha \mathcal{Q}\theta^j) =: (\varphi_0, \dots, \varphi_{n-1})$ is increasing, where $\varphi_i := \beta_{i+j} + \alpha - 2\pi$ for $i \in \overline{0, n-j-1}$, and $\varphi_i := \beta_{i+j-n} + \alpha$ for $i \in \overline{n-j, n-1}$. Thus, the cyclic permutation $R_\alpha \mathcal{P}\theta^m = R_\alpha \mathcal{P}\theta^{k+j} = R_\alpha \mathcal{Q}\theta^j$ of polygon $R_\alpha \mathcal{P}$ is increasing, where $m := k+j$ if $k+j < n$ and $m := k+j-n$ if $k+j \geqslant n$. Thus, $R_\alpha \mathcal{P}$ is c-increasing.

The preservation of the c-decreasing property under any rotation is verified quite similarly.

The other claims stated in Proposition 1.11 are only easier to check. $\square$

*Proof of Theorem 1.12.* Let $\mathcal{P} = (V_0, \dots, V_{n-1})$ be an $n$-gon with $n \geqslant 3$, vertices $V_i =: (x_i, y_i)$, and argument $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$. In view of Proposition 1.11, the rotation $R_{2\pi-\alpha_0}$ and any homothetical transformation will preserve both the convexity and c-monotonicity properties of $\mathcal{P}$. Therefore, assume without loss of generality (w.l.o.g.) that $\alpha_0 = 0$ and, moreover, $V_0 = (0, 0)$ and $V_1 = (1, 0)$.

**"If"** When proving this part, assume w.l.o.g. that $\mathcal{P}$ is c-increasing, that is, $\alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1}$. $\big($Indeed, in view of Proposition 1.11, the reflection transformation $R$ will preserve the convexity property of $\mathcal{P}$ and interchange the property of $\mathcal{P}$ being c-increasing with it being c-decreasing; also, $R$ will preserve the property $\alpha_0 = 0.\big)$ Then the conditions $\alpha_0 = 0$ and $\alpha_i \in [0, 2\pi)$ $\forall i$ imply that $k = 0$ and $\alpha_0 = 0 < \dots < \alpha_{n-1}$; that is, the sequence $\arg \mathcal{P}$ is increasing.

Hence, inequality $\alpha_1 \geqslant \pi$ would imply $\alpha_i \in (\pi, 2\pi)$ for all $i \in \overline{2, n-1}$. Hence and because $n \geqslant 3$, one would have $0 = y_1 \geqslant y_2 > y_3 > \dots > y_n = y_0 = 0$, and at least one inequality here is strict (since $n \geqslant 3$), which is a contradiction.

The case $\alpha_1 < \pi$ is similar. In this case, $y_2 > 0$. To obtain a contradiction, suppose that the set $L := \{i \in \overline{2, n-1} \colon y_i \leqslant 0\}$ is non-empty and then let $\ell := \min L$, so that $\ell \in \overline{3, n-1}$, $y_{\ell-1} > 0$, and $y_\ell \leqslant 0$. Then $\alpha_{\ell-1} \in (\pi, 2\pi)$. Hence and because the sequence $\arg \mathcal{P}$ is increasing, one has $\alpha_i \in (\pi, 2\pi)$ for all $i \in \overline{\ell-1, n-1}$. Therefore, $0 \geqslant y_\ell > \dots > y_n = y_0 = 0$, which is a contradiction. This contradiction means that $L = \emptyset$, so that $y_i > 0$ for all $i \in \overline{2, n-1}$; that is, according to [7, Definition 2.4], the polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is strictly to one side of its edge $[V_0, V_1]$.

Similarly it is proved that $\mathcal{P}$ is strictly to one side of any other one of its edges; that is, $\mathcal{P}$ is strictly to-one-side. To complete the proof of the "if" part of Theorem 1.12, it remains to refer to [7, Lemmas 2.6 and 2.11].

**"Only if"** Here is assumed that polygon $\mathcal{P}$ is strictly convex. Again w.l.o.g. one has $\alpha_0 = 0$. Also, by Remark 1.13, w.l.o.g. $n \geqslant 4$. Again by the "reflection" part of Proposition 1.11, w.l.o.g. $y_2 \geqslant 0$. Moreover, because of the strictness of $\mathcal{P}$ and the assumptions $V_0 = (0, 0)$ and $V_1 = (1, 0)$, one actually has $y_2 > 0$, so that $\alpha_0 = 0 < \alpha_1 < \pi$ and $\Delta_{0,1,2} = y_2 > 0$. So, the strict convexity of $\mathcal{P}$ and Remark 1.7 yield $\Delta_{0,1,i} = y_i > 0$ for all $i \in \overline{2, n-1}$. The strictness of $\mathcal{P}$ also implies that all the values $\alpha_0, \dots, \alpha_{n-1}$ are distinct from one another.

Thus, it suffices to show that $\alpha_i \leqslant \alpha_{i+1}$ for all $i \in \overline{1, n-2}$. Suppose the contrary, that $\alpha_i > \alpha_{i+1}$ for some $i \in \overline{1, n-2}$. Consider separately the following three cases.

*Case 1:* $i = 1$. Then $\alpha_1 > \alpha_2$. By (7), this implies that $\Delta_{1,2,3} \leqslant 0$ or $y_2 - y_1 \leqslant 0 \leqslant y_3 - y_2$; but $y_2 - y_1 = y_2 > 0$, so that one must have $\Delta_{1,2,3} \leqslant 0$; now inequalities $\Delta_{1,2,3} \leqslant 0$ and $\Delta_{0,1,2} > 0$ contradict Remark 1.7.

*Case 2:* $i = n-2$. Then $\alpha_{n-2} > \alpha_{n-1}$. This case is quite similar to Case 1. Indeed, by (7), here one has $\Delta_{0,n-2,n-1} = \Delta_{n-2,n-1,0} \leqslant 0$ or $0 \leqslant y_0 - y_{n-1}$; but $y_0 - y_{n-1} = -y_{n-1} < 0$, so that $\Delta_{0,n-2,n-1} \leqslant 0$; now inequalities $\Delta_{0,n-2,n-1} \leqslant 0$ and $\Delta_{0,1,2} > 0$ contradict Remark 1.7.

*Case 3:* $i \in \overline{2, n-3}$ and $\alpha_i > \alpha_{i+1}$. Then the 5-gon $\mathcal{Q} := (V_0, V_1, V_i, V_{i+1}, V_{i+2})$ is a sub-polygon of $\mathcal{P}$, so that $\mathcal{Q}$ is strictly convex, by [6, Corollary 1.17]. On the other hand, $\arg \mathcal{Q} = (\alpha_0, \beta, \alpha_i, \alpha_{i+1}, \gamma)$, for some real numbers $\beta$ and $\gamma$. Thus, w.l.o.g. $\mathcal{P} = \mathcal{Q}$, $n = 5$, and so, one has all of the following: $\mathcal{P} = (V_0, V_1, V_2, V_3, V_4)$; $i = 2$; $\alpha_2 > \alpha_3$; and $\Delta_{0,1,i} = y_i > 0$ for all $i \in \overline{2, 4}$. By Remark 1.7, one now also sees that the determinants $\Delta_{2,3,4}$, $\Delta_{0,2,3}$, and $\Delta_{0,3,4}$ are all strictly positive as well. Therefore, the condition $\alpha_2 > \alpha_3$ and implication (7) yield $y_3 - y_2 \leqslant 0 \leqslant y_4 - y_3$. One can verify the identity

$$(y_4 - y_3)\,\Delta_{0,2,3} + (y_2 - y_3)\,\Delta_{0,3,4} + \Delta_{2,3,4}\,\Delta_{0,1,3} = 0,$$

whose left-hand side is strictly positive, since all the determinants $\Delta_{\cdot,\cdot,\cdot}$ in this identity are strictly positive and because of the condition $y_3 - y_2 \leqslant 0 \leqslant y_4 - y_3$. Thus, one obtains a contradiction.

The proof of the "only if" part and thus of entire Theorem 1.12 is now complete. $\square$

Let $U \_\_ V \_\_ W \_\_ \dots$ (respectively, $U, \widehat{V, W}, \dots$) mean that $U, V, W, \dots$ are collinear (respectively, non-collinear) points on the plane.

**Lemma 2.1.** [Cf. [7, Lemma 2.7].] *For any choice of $\alpha$, $\beta$, $i$, and $j$ in $\overline{0, n-1}$, points $V_\alpha$ and $V_\beta$ are to one side of of $[V_i, V_j]$ if and only if $\Delta_{\alpha,i,j}\,\Delta_{\beta,i,j} \geqslant 0$, where $\Delta_{\alpha,i,j}$ are given by* (2)*; for an exact definition of "to one side", see [7, Definition 2.4].*

*Proof of Lemma 2.1.* This proof can be done quite similarly to that of [7, Lemma 2.7]. Alternatively and more simply, Lemma 2.1 can be easily deduced from [7, Lemma 2.7] by observing that $V_\alpha \_\_ V_i \_\_ V_j$ if and only if $\Delta_{\alpha,i,j} = 0$. $\square$

*Proof of Proposition 1.14.* W.l.o.g., $n \geqslant 4$.

**(I) $\Longrightarrow$ (II)** Here it is assumed that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is ordinary and locally-strictly convex. By the c-shift invariance (Remark 1.10), at this point it suffices to show that $\widehat{V_0, V_1, V_i}$ for each $i \in \overline{2, n-1}$ or just only for each $i \in \overline{3, n-1}$, because $\widehat{V_0, V_1, V_2}$ follows from $\mathcal{P}$ being locally-strict.

To obtain a contradiction, assume first that $V_0 \_\_ V_1 \_\_ V_3$. Since $\mathcal{P}$ is locally-strictly convex, one has $\widehat{V_0, V_1, V_2}$, so that, by the affine invariance (Proposition 1.11), w.l.o.g. $V_0 = (0, 0)$, $V_1 = (1, 0)$, and $V_2 = (1, 1)$. Then $V_0 \_\_ V_1 \_\_ V_3$ implies that $V_3 = (x_3, 0)$ for some real $x_3$. Since $\mathcal{P}$ is ordinary, one has $x_3 \notin \{0, 1\}$. Hence, there are only the following three cases to consider at this point:

*Case 1:* $x_3 > 1$. Then $\Delta_{1,2,0}\,\Delta_{1,2,3} = 1 - x_3 < 0$. By Lemma 2.1, this means that $V_0$ and $V_3$ are not on one side of $[V_1, V_2]$; by [6, Lemma 2.3], this contradicts the

convexity of polygon $\mathcal{P}$.
*Case 2:* $0 < x_3 < 1$. Then $\Delta_{2,3,0}\,\Delta_{2,3,1} = x_3(x_3-1) < 0$, so that $V_0$ and $V_1$ are not on one side of $[V_2, V_3]$, which contradicts the convexity of polygon $\mathcal{P}$.
*Case 3:* $x_3 < 0$. Then $V_{n-1} = (x_{n-1}, y_{n-1})$ for some real $x_{n-1}$ and $y_{n-1}$ such that $y_{n-1} \neq 0$ – because $\mathcal{P}$ is locally strict and hence $\widehat{V_{n-1}, V_0, V_1}$. Hence, $\Delta_{n-1,0,1}\,\Delta_{n-1,0,3} = y_{n-1}^2 x_3 < 0$, so that $V_0$ and $V_3$ are not on one side of edge $[V_{n-1}, V_0]$, which contradicts the convexity of polygon $\mathcal{P}$.

Thus, in all cases the assumption $V_0 \_ V_1 \_ V_3$ leads to a contradiction, so that one has $\widehat{V_0, V_1, V_3}$. Similarly one proves that $\widehat{V_1, V_3, V_4}$ (for $n = 4$ this has been already proved, for then $V_4 = V_0$). That is, the sub-polygon $\mathcal{P}^{(2)} := (V_0, V_1, V_3, \dots, V_{n-1})$ is locally strict. Also, by Theorem 1.4, $\mathcal{P}^{(2)}$ inherits the property of $\mathcal{P}$ of being ordinarily convex. Now it follows by induction that $\mathcal{P}^{(2)}$ is quasi-strict, so that indeed $\widehat{V_0, V_1, V_i}$ for each $i \in \overline{3, n-1}$. This proves (I) $\Longrightarrow$ (II).

**(II) $\Longrightarrow$ (III)** This is immediate from [7, Lemma 2.11].

**(III) $\Longrightarrow$ (I)** This is immediate from [7, Lemmas 2.11 and 2.5]. □

Introduce the "direction" equivalence on $\mathbb{R}^2 \setminus \{\vec{0}\}$ defined by the formula

$$\vec{u} \upuparrows \vec{v} \overset{\text{def}}{\iff} \vec{v} = \lambda\vec{u} \text{ for some } \lambda > 0,$$

for any $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2 \setminus \{\vec{0}\}$. Note that

$$\vec{u} \upuparrows \vec{v} \implies (\vec{u} + \vec{v}) \upuparrows \vec{u}. \tag{8}$$

**Lemma 2.2.** *For any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2 \setminus \{\vec{0}\}$, one has $\arg\vec{u} = \arg\vec{v}$ iff $\vec{u} \upuparrows \vec{v}$.*

*Proof of Lemma 2.2.* This folows immediately from (4). □

**Lemma 2.3.** *Suppose that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is locally-ordinary and for some $j$ and $k$ in $\overline{0, n-1}$ such that $j \leqslant k$ one has $\alpha_j = \cdots = \alpha_k$, where $(\alpha_0, \dots, \alpha_{n-1}) := \arg\mathcal{P}$. Then $\overrightarrow{V_jV_{k+1}} \neq \vec{0}$ and*

(I) $\arg\overrightarrow{V_jV_{k+1}} = \arg\overrightarrow{V_iV_{i+1}}$ *for all $i$ in $\overline{j,k}$;*
(II) $[V_j, V_{k+1}] = \bigcup_{i=j}^{k}[V_i, V_{i+1}]$*;*
(III) $[V_i, V_{i+1}] \cap [V_t, V_{t+1}] = \emptyset$ *for all $i$ and $t$ in $\overline{j,k}$ such that $i \neq t$.*

*Proof of Lemma 2.3.* By Lemma 2.2, the condition $\alpha_j = \cdots = \alpha_k$ implies that for each $i$ in $\overline{0, n-2}$ one has $\overrightarrow{V_{i+1}V_{i+2}} \upuparrows \overrightarrow{V_iV_{i+1}}$, so that, in view of (8), $\overrightarrow{V_jV_{k+1}} \upuparrows \overrightarrow{V_iV_{i+1}}$, which is equivalent to (I), again by Lemma 2.2.

In particular, these observations imply that the points $V_j, \dots, V_{k+1}$ lie on the same straight line, say $\ell$. Let $A$ be any non-singular affine mapping of $\ell$ onto $\mathbb{R}$ such that $AV_j < AV_{j+1}$ (such a mapping exists because $\mathcal{P}$ is locally-ordinary and hence $V_j \neq V_{j+1}$). Then the condition $\overrightarrow{V_{i+1}V_{i+2}} \upuparrows \overrightarrow{V_iV_{i+1}}$ implies $V_{i+1} \in (V_i, V_{i+2})$, whence $AV_{i+1} \in (AV_i, AV_{i+2})$, again for each $i$ in $\overline{0, n-2}$, so that $AV_j < \cdots <$

$AV_{k+1}$, $[AV_j, AV_{k+1}] = \bigcup_{i=j}^{k} [AV_i, AV_{i+1}]$, and $[AV_i, AV_{i+1}] \cap [AV_t, AV_{t+1}] = \emptyset$ for all distinct $i$ and $t$ in $\overline{j,k}$. It remains to note that $[AU, AV] = A[U, V]$ for any points $U$ and $V$ on $\ell$ and recall that $A$ is non-singular and hence one-to-one. $\square$

**Lemma 2.4.** *Any strictly convex $n$-gon with $n \geqslant 3$ is simple.*

*Proof of Lemma 2.4.* Suppose that an $n$-gon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is strictly convex and take any $i$ and $j$ in $\overline{0, n-1}$ such that $i < j$. We have then to show that $[V_i, V_{i+1}) \cap [V_j, V_{j+1}) = \emptyset$. By the c-shift invariance (Remark 1.10), w.l.o.g. $i = 0$ and then $j \in \overline{1, n-1}$. By Remark 1.3, $\mathcal{P}$ is ordinary, so that $V_1 \neq V_0$. Let $\ell := V_0V_1$, the straight line through $V_0$ and $V_1$. By [7, Lemma 2.6], there is an open half-plane $H$ such that $\partial H = \ell$ and $\{V_2, \dots, V_{n-1}\} \subset H$. Then (with $i = 0$)

$$[V_i, V_{i+1}) \cap [V_j, V_{j+1}) = [V_0, V_1) \cap (\ell \cap [V_j, V_{j+1})) \subseteq [V_0, V_1) \cap \{V_j\} = \emptyset; \tag{9}$$

the set inclusion in (9) follows because $(V_j, V_{j+1}) = \operatorname{ri}[V_j, V_{j+1}] \subseteq \operatorname{interior}(\ell \cup H) = H \subseteq \mathbb{R}^2 \setminus \ell$; the last equality in (9) is trivial for $j = 1$ and takes place for $j \in \overline{2, n-1}$ because then $[V_0, V_1) \cap \{V_j\} \subseteq \ell \cap H = \emptyset$. $\square$

**Lemma 2.5.** *Suppose that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is ordinary and locally-simple. Suppose also that for some $j$ and $k$ in $\overline{0, n-1}$ one has $V_j \text{--} \cdots \text{--} V_{k+1}$. Then $\alpha_j = \cdots = \alpha_k$, where $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$.*

*Proof of Lemma 2.5.* Take any $i$ in $\overline{j, k-1}$. Then $\overrightarrow{V_{i+1}V_{i+2}} = \mu_i \overrightarrow{V_iV_{i+1}}$ for some $\mu_i \neq 0$, given the conditions that $V_j \text{--} \cdots \text{--} V_{k+1}$ and $\mathcal{P}$ is ordinary. Then inequality $\mu_i < 0$ would imply $[V_i, V_{i+1}) \cap [V_{i+1}, V_{i+2}) \supseteq (\tilde{V}_i, V_{i+1}) \neq \emptyset$, since $\mathcal{P}$ is ordinary, where $\tilde{V}_i := V_i$ if $\mu_i \leqslant -1$ and $\tilde{V}_i := V_{i+2}$ if $-1 < \mu_i < 0$; so, this would contradict the condition that $\mathcal{P}$ is locally-simple.

It follows that $\mu_i > 0$ and hence $\overrightarrow{V_{i+1}V_{i+2}} \upuparrows \overrightarrow{V_iV_{i+1}}$ for all $i$ in $\overline{j, k-1}$. It remains to refer to Lemma 2.2. $\square$

*Proof of Theorem 1.15.*

**(I) $\implies$ (III)** Here it is assumed that an $n$-gon $\mathcal{P} = (V_0, \dots, V_{n-1})$, with $n \geqslant 3$ and $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$, is ordinary and locally-simply convex. We have to prove that $\mathcal{P}$ is c-monotone. Introduce the set

$$M := \{i \in \overline{0, n} \colon \widehat{V_{i-1}, V_i, V_{i+1}}\},$$

where $V_{-1} := V_{n-1}$ and $V_{n+1} := V_1$.

Note that $M \neq \emptyset$. Indeed, otherwise one would have $\alpha_0 = \cdots = \alpha_{n-1}$, by Lemma 2.5; then Lemma 2.3(II) would imply $[V_0, V_1] \subseteq [V_0, V_n] = \{V_0\}$, which would contradict the condition that $\mathcal{P}$ is ordinary.

Therefore, $\dim \mathcal{P} = 2$. Also, by the c-shift invariance (Remark 1.10), w.l.o.g. $0 \in M$ and hence (by the definition of $M$) one has $n \in M$, so that

$$M = \{j_0, \dots, j_m\}$$

for some $m \in \overline{0, n-1}$ and integers $j_0, \ldots, j_m$ such that $0 = j_0 < \cdots < j_m = n$, and, moreover,

$$
\begin{aligned}
&V_{n-1}, \widehat{V_0}, V_1, \qquad && V_0 = V_{j_0} \text{--} \cdots \text{--} V_{j_1};\\
&V_{j_1-1}, \widehat{V_{j_1}}, V_{j_1+1}, && V_{j_1} \text{--} \cdots \text{--} V_{j_2};\\
&\quad\vdots\\
&V_{j_{m-1}-1}, \widehat{V_{j_{m-1}}}, V_{j_{m-1}+1}, && V_{j_{m-1}} \text{--} \cdots \text{--} V_{j_m} = V_n = V_0.
\end{aligned}
\tag{10}
$$

By Theorem 1.4, the sub-polygon

$$
\mathcal{Q} := (U_0, \ldots, U_{m-1}) := (V_{j_0}, \ldots, V_{j_{m-1}}) \tag{11}
$$

of the ordinarily convex polygon $\mathcal{P}$ is ordinarily convex as well.

Note also that $\mathcal{Q}$ is locally-strict. To check this, in view of the c-shift invariance (Remark 1.10) it suffices to show that $U_0, \widehat{U_1}, U_2$ or, equivalently, $V_{j_0}, \widehat{V_{j_1}}, V_{j_2}$. But this follows by the the ordinariness of $\mathcal{P}$ and construction of $\mathcal{Q}$ (whereby $V_{j_1-1}, \widehat{V_{j_1}}, V_{j_1+1}$, $V_{j_0} \text{--} V_{j_1-1} \text{--} V_{j_1}$, and $V_{j_1} \text{--} V_{j_1+1} \text{--} V_{j_2}$).

Hence, by Proposition 1.14, $\mathcal{Q}$ is strictly convex. Now, in view of Theorem 1.12, $\mathcal{Q}$ is c-strictly monotone. Applying (if necessary) the reflection transformation $R$ and a cyclic shift $\theta^k$ and referring to Proposition 1.11 and Remark 1.10, assume w.l.o.g. that polygon $\mathcal{Q}$ is increasing:

$$
\arg \overrightarrow{V_{j_0} V_{j_1}} < \cdots < \arg \overrightarrow{V_{j_{m-1}} V_{j_m}}. \tag{12}
$$

To complete the proof of implication (I) $\Longrightarrow$ (III) of Theorem 1.15, it remains to refer to (10), Lemma 2.5, and (12), whereby

$$
\begin{aligned}
\alpha_0 = {} & \quad \alpha_{j_0} = \ldots = \alpha_{j_1-1} = \arg \overrightarrow{V_{j_0} V_{j_1}}\\
< \arg \overrightarrow{V_{j_1} V_{j_2}} = {} & \quad \alpha_{j_1} = \ldots = \alpha_{j_2-1}\\
& \vdots\\
< \arg \overrightarrow{V_{j_{m-1}} V_{j_m}} = {} & \ \alpha_{j_{m-1}} = \ldots = \alpha_{j_m-1} = \alpha_{n-1}.
\end{aligned}
$$

**(III) $\Longrightarrow$ (II)** Here it is assumed that a polygon $\mathcal{P} = (V_0, \ldots, V_{n-1})$ with $(\alpha_0, \ldots, \alpha_{n-1}) := \arg \mathcal{P}$ is c-monotone. We have to prove that $\mathcal{P}$ is simply convex. Applying the reflection transformation $R$ and a cyclic shift $\theta^k$, w.l.o.g. polygon $\mathcal{P}$ is non-decreasing:

$$
\alpha_0 \leqslant \ldots \leqslant \alpha_{n-1}.
$$

That is,

$$
\alpha_{j_0} = \cdots = \alpha_{j_1-1} < \cdots < \alpha_{j_{m-1}} = \cdots = \alpha_{j_m-1} \tag{13}
$$

for some natural $m$ and integer $j_0, \ldots, j_m$ such that

$$
0 = j_0 < \cdots < j_m = n. \tag{14}
$$

Define polygon $\mathcal{Q}$ again by (11). Then, in view of Lemma 2.3(I), one has $\arg \mathcal{Q} = (\alpha_{j_0}, \dots, \alpha_{j_{m-1}})$, so that $\mathcal{Q}$ is increasing and hence, by Theorem 1.12, strictly convex.

By Lemma 2.3(II), one has

$$[U_p, U_{p+1}] = [V_{j_p}, V_{j_{p+1}}] = \bigcup_{i=j_p}^{j_{p+1}-1} [V_i, V_{i+1}]$$

for all $p \in \overline{0, m-1}$, so that $\operatorname{edg} \mathcal{Q} = \operatorname{edg} \mathcal{P}$. Therefore, moreover,

$$\{V_0, \dots, V_{n-1}\} \subseteq \operatorname{edg} \mathcal{P} = \operatorname{edg} \mathcal{Q} \subseteq \operatorname{conv} \mathcal{Q} \subseteq \operatorname{conv} \mathcal{P},$$

whence $\operatorname{conv} \mathcal{P} = \operatorname{conv} \mathcal{Q}$. Thus, the convexity of $\mathcal{Q}$ implies the convexity of $\mathcal{P}$ and $\dim \mathcal{Q} = \dim \mathcal{P} = 2$, so that $m \geqslant 3$.

Let us now show that $\mathcal{P}$ simple. That is, let us take any $i$ and $j$ in $\overline{0, n-1}$ such that $i \neq j$ and show that $[V_i, V_{i+1}) \cap [V_j, V_{j+1}) = \emptyset$. By (14), there exist $p$ and $q$ in $\overline{0, m-1}$ such that $i \in J_p$ and $j \in J_q$, where $J_s := \overline{j_s, j_{s+1} - 1}$ for all $s \in \overline{0, m-1}$.

If $p = q$ then $[V_i, V_{i+1}) \cap [V_j, V_{j+1}) = \emptyset$ follows from Lemma 2.3(III) and (13).

If $p \neq q$ then, by Lemma 2.3(II),

$$[V_i, V_{i+1}) \cap [V_j, V_{j+1}) \subseteq [U_p, U_{p+1}) \cap [U_q, U_{q+1}) = \emptyset,$$

the latter equality taking place since $m \geqslant 3$ and the $m$-gon $\mathcal{Q}$ is strictly convex and hence (by Lemma 2.4) simple.

This completes the proof of implication (III) $\Longrightarrow$ (II) of Theorem 1.15.

**(II) $\Longrightarrow$ (I)** This implication is the easiest to prove. Indeed, if a polygon $\mathcal{P}$ is simple then it is (trivially) locally-simple and (by Remark 1.3) ordinary. This completes the proof of Theorem 1.15. $\square$

## Literature


[1] F. S. Hill, Jr. *The pleasures of "perp dot" products.* In Graphics Gems IV, P. S. Heckbert, ed., Academic Press, London, 138–148, 1994.

[2] K. Mehlhorn and St. Näher, The LEDA Platform of Combinatorial and Geometric Computing, Cambridge University Press, 1999. http://www.mpi-sb.mpg.de∼mehlhorn/LEDAbook.html.

[3] Mehlhorn, K. et al., *Checking Geometric Programs or Verification of Geometric Structures.* Computational Geometry, 12(1-2):85-103, 1999.

[4] B. M. E. Moret and H. D. Shapiro, *Algorithms from N to NP. Volume 1. Design and Efficiency*, The Benjamin/Cummings Publishing Company, 1991.

[5] I. Pinelis, Cyclic polygons with given edge lengths: existence and uniqueness. *J. Geom.* **82** (2005), no. 1-2, 156–171.

[6] I. Pinelis, *Convexity of sub-polygons of convex polygons.* Preprint (2006), `http://arxiv.org/abs/math.GM/0609698`.

[7] Pinelis, I., *Polygon Convexity: A Minimal $O(n)$ Test.* Preprint (2006), `http://arxiv.org/abs/cs.CG/0609141`.

[8] Pinelis, I., *An order-refined and generalized version of the Erdös-Szekeres theorem on convex polygons.* Preprint (2006), `http://arxiv.org/abs/math.CO/0611802`.

[9] R. T. Rockafellar, *Convex Analysis*, Princeton University Press 1970.
[10] E. W. Weisstein. Convex polygon. From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/ConvexPolygon.html` .
[11] I. M. Yaglom and V. G. Boltyanskiĭ, *Convex Figures*, Holt, Rinehart and Winston 1961.

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